Solve the initial value problem $(\theta\cos\theta)r' + (\cos\theta + \theta\sin\theta)r = 1, \quad r(\frac{\pi}{4}) = \sqrt{2}$.

SCORE: _____ / 30 PTS

ALL ITEMS 3 POINTS UNLESS OTHERWISE NOTED

$$\frac{dr}{d\theta} + \left(\frac{1}{\theta} + \tan\theta\right) r = \frac{1}{\theta\cos\theta}$$

$$\mu = e^{\int\left(\frac{1}{\theta} + \tan\theta\right)d\theta} = e^{\ln|\theta| + \ln|\sec\theta|} = \theta\sec\theta$$

$$(\theta\sec\theta)\frac{dr}{d\theta} + (\sec\theta + \theta\sec\theta\tan\theta)r = \sec^2\theta$$

CHECKPOINT: $\frac{d}{d\theta}(\theta\sec\theta) = \sec\theta + \theta\sec\theta\tan\theta$ ✓

$$(\theta\sec\theta)r = \int\sec^2\theta\, d\theta + C$$

$$= \tan\theta + C$$

$$r = \frac{\sin\theta}{\theta} + \frac{C\cos\theta}{\theta} = \frac{\sin\theta + C\cos\theta}{\theta}$$

$$\cancel{\sqrt{2}} = \frac{\frac{\sqrt{2}}{2} + C\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \frac{2\cancel{\sqrt{2}} + 2\cancel{\sqrt{2}}C}{\pi}$$

$$\pi = 2 + 2C$$

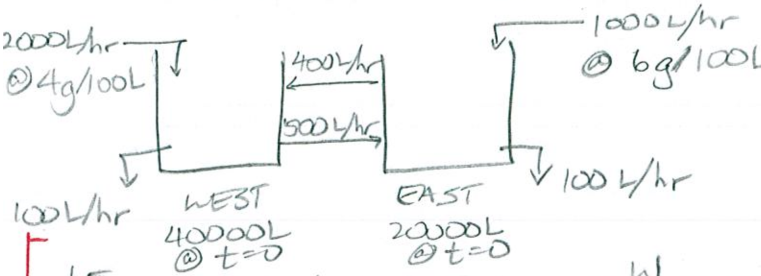
$$C = \frac{\pi}{2} - 1 \quad (4)$$

$$r = \frac{\sin\theta + (\frac{\pi}{2} - 1)\cos\theta}{\theta}$$

$$r = \frac{2\sin\theta + (\pi - 2)\cos\theta}{2\theta} \quad (2)$$

A factory sits near two ponds, and is dumping its wastewater into both ponds. Every hour, the factory dumps 1000 liters of wastewater containing 6 grams of radioactive material per 100 liters into the east pond, and 2000 liters of wastewater containing 4 grams of radioactive material per 100 liters into the west pond. In addition, the two ponds are connected so that content from each pond seeps into the other. Each hour, 400 liters of the east pond's content seeps into the west pond, and 500 liters of the west pond's content seeps into the east pond. An additional 100 liters of each pond also drains away each hour. **SCORE: _____ / 25 PTS**

If the east pond originally contained 20000 liters, the west pond originally contained 40000 liters, and both ponds were well-mixed at all times, write, **BUT DO NOT SOLVE**, a system of differential equations for the amount of radioactive material in each pond.



VOLUME OF EAST POND

$= 20000 + (1000 + 500 - 400 - 100)t$

$= 20000 + 1000t$

VOLUME OF WEST POND

$= 40000 + (2000 + 400 - 500 - 100)t$

$= 40000 + 1800t$

$$\frac{dE}{dt} = 1000 * \frac{6}{100} + 500 * \frac{W}{40000+1800t}$$ (4)

(1½)

(2)

$$- (400+100) * \frac{E}{20000+1000t}$$ (6)

$$\frac{dW}{dt} = 2000 * \frac{4}{100} + 400 * \frac{E}{20000+1000t} - (500+100) * \frac{W}{40000+1800t}$$ (4) (5)

(1½)

$$\frac{dE}{dt} = 60 + \frac{5W}{400+18t} - \frac{E}{40+2t}, \quad \frac{dW}{dt} = 80 + \frac{2E}{100+5t} - \frac{3W}{200+9t}$$ (1) (1)

Commensalism is a relationship between two populations in which interaction between the two populations helps one population (the commensal) to grow, but has no effect on the other population (the host) at all. (An example would be insects that eat the dead skin fragments that have fallen from mammals.) The host population grows at a rate proportional to its existing population, regardless of the presence of the commensal; whereas, the commensal population living without the host would die off at a rate proportional to its existing population, but interactions with the host provide growth of the population. (In the discussion of the predator-prey model, we considered how interactions affected the rate of population change.) **SCORE: _____ / 15 PTS**

[a] Write, **BUT DO NOT SOLVE**, a system of differential equations for the populations for a commensal (C) and a host (H).
NOTE: State clearly what your variables represent.
Symbolic constants that appear in your answer must represent positive numbers.

$$\frac{dC}{dt} = -aC + bCH$$ (7)

$$\frac{dH}{dt} = kH$$ (3)

[b] Is there a non-trivial equilibrium for a commensal relationship ? That is, an equilibrium other than both populations equal 0 ?
If yes, find it. If no, justify your answer **algebraically**. (Verbal non-algebraic justifications earn no credit.)

$$\frac{dH}{dt} = 0 \rightarrow kH = 0 \rightarrow H = 0$$ (2)

$$\frac{dC}{dt} = 0 \rightarrow -aC + bC(0) = 0 \rightarrow C = 0$$ (2)

(1) ONLY THE TRIVIAL EQUILIBRIUM $H = C = 0$

Solve the differential equation $z' = \dfrac{z^2}{t^2 - 2z^2}$.

SCORE: _____ / 40 PTS

ALL ITEMS 3 POINTS UNLESS OTHERWISE NOTED

$$\underbrace{(t^2-2z^2)}_{M}dz - \underbrace{z^2}_{N}dt = 0$$

$$M(kt,kz) = (kt)^2 - 2(kz)^2 = k^2(t^2-2z^2) = k^2M(t,z)$$
$$N(kt,kz) = -(kz)^2 = k^2(-z^2) = k^2N(t,z)$$

BOTH HOMOGENEOUS DEGREE 2

LET $t = vz$

$dt = v\,dz + z\,dv$

$$(v^2\cancel{z^2} - 2\cancel{z^2})dz - \cancel{z^2}(v\,dz + z\,dv) = 0$$

$$(v^2 - v - 2)dz - z\,dv = 0$$

$$\int \frac{1}{z}dz = \int \frac{1}{v^2-v-2}dv = \int\left(\frac{\frac{1}{3}}{v-2} + \frac{-\frac{1}{3}}{v+1}\right)dv \quad 8$$

$$C + \ln|z| = \frac{1}{3}\ln|v-2| - \frac{1}{3}\ln|v+1|$$

$$C + 3\ln|z| = \ln\left|\frac{v-2}{v+1}\right|$$

$$Cz^3 = \frac{v-2}{v+1} = \frac{\frac{t}{z}-2}{\frac{t}{z}+1} = \frac{t-2z}{t+z}$$

$$Cz^3(t+z) = t - 2z \quad 2$$

Solve the initial value problem $(y-e^y)\frac{dy}{dx}+2e^y=y^2+2e^x, \quad y(0)=0$.

SCORE: _____ / 40 PTS

ALL ITEMS 3½ POINTS UNLESS OTHERWISE NOTED

$$\underbrace{(y-e^y)}_{M}dy+\underbrace{(2e^y-y^2-2e^x)}_{N}dx=0$$

$$\frac{N_y-M_x}{M} = \text{6 } \frac{2e^y-2y-0}{y-e^y} = -2 \quad \text{FUNCTION OF ONLY } x$$

$$\mu = e^{\int -2\,dx} = e^{-2x}$$

$$\underbrace{(y-e^y)e^{-2x}}_{M}dy+\underbrace{(2e^ye^{-2x}-y^2e^{-2x}-2e^{-x})}_{N}dx=0$$

$$M_x = -2(y-e^y)e^{-2x} = -2ye^{-2x}+2e^ye^{-2x} = N_y$$

CHECKPOINT: EXACT

$$f = \int (2e^ye^{-2x}-y^2e^{-2x}-2e^{-x})\,dx$$

$$= -e^ye^{-2x}+\tfrac{1}{2}y^2e^{-2x}+2e^{-x}+C(y)$$

$$f_y = -e^ye^{-2x}+ye^{-2x}+C'(y) = (y-e^y)e^{-2x}$$

CHECKPOINT: ONLY y

$$C'(y)=0 \quad \text{2}$$

$$C(y)=0 \quad \text{2}$$

$$-e^ye^{-2x}+\tfrac{1}{2}y^2e^{-2x}+2e^{-x}=C$$

$$-1+0+2=C=1$$

$$-e^ye^{-2x}+\tfrac{1}{2}y^2e^{-2x}+2e^{-x}=1 \quad \text{2}$$